Biostats VHM 801 Course
Fall 2016, Atlantic Veterinary College, PEI
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Index of 6-L

(minor revision on page 13)

Practical Information

Today's lecture:

- another key concept from statistical inference:
  - test (significance, P-value),[1]
- inference for one sample (continuous data):[2]
  - with or without assumption assumption of known σ,
  - a new distribution: the t-distribution,
- postpone material on sample size and power (Session 9),
- Summary Worksheets from S: Chapters 1 and 7.

Home assignment:

- deadline is anytime today (tonight) in my mailbox/office or by electronic submission on Moodle,
- returned (with marks, comments and solution) to you next Thursday; possibly also brief discussion in class,
- 2nd home assignment in $1\frac{1}{2}$ weeks (Monday 17/10).

Schedule:

- midterm scheduled for Friday 28/10, 1-2pm,
- lab review for Session 5: today 1-2pm (again in 351S),
- next lab: tomorrow (Friday 7/10), 1-4pm.

[1] PSLS 3e: Chapter 14-15 (parts); S: Chapter 8; IPS 7e: Sections 6.2-3.
[2] PSLS 3e: Chapter 17; S: Chapters 7-8; IPS 7e: Section 7.1.

Two Examples of Test Problems

Example I: Testing of taste (example not in textbooks):

- aim: compare two brands of wine (beer, milk, cheese...),
- "duo-trio test" with one subject (person):
  - two anonymized samples, one of each brand,
  - third sample of known type,
  - subject may taste all 3 samples, as (s)he likes,
  - task: determine brand of two unknown samples,
- repeating the experiment, subject scores x out of n (e.g., 6 out of 8) correctly – how to determine if result has not occurred by chance ("luck")?
- statistical problem because of randomness associated with "guessing" (even if qualified guessing).

Example II: Laboratory analysis of active ingredient in specimens: (Exercise 14.5 in PSLS 3e)

- data from 3 analyses of one specimen:
  0.8403, 0.8363, 0.8447 (in g/l),
- aim: evaluate producer's specified content of 0.86 g/l,
- statistical problem because of random measurement errors in laboratory.

Introduction to Statistical Testing

Consider the "duo-trio" testing problem, and let X denote the number of "successes" for one subject in 8 trials.

- binomial setting $\Rightarrow$ $X \sim$ binomial distrib. $\mathrm{B}(8, p)$,
- if guessing, the probability p in each trial must be $p{=}0.5$ — state this as our *null hypothesis* H_0: $p{=}0.5$,
- under H_0: $X \sim \mathrm{B}(8, 0.5)$:

x	0	1	2	3	4	5	6	7	8
$\mathrm{P}(X{=}x)$	0.004	0.03	0.11	0.22	0.27	0.22	0.11	0.03	0.004

- alternatively to H_0 we must have $p > 0.5$ (unless subject messes up the experiment) − state this as our *alternative hypothesis* H_a: $p > 0.5$,

If subject gets all trials right ($X = 8$):

* probability of event happened by chance: $P = 0.004$,
* by low P-value, we have little confidence in H_0 (because observed event unlikely to happen if H_0 was true) $\Rightarrow$ *reject* H_0 and prefer H_a, (but H_0 could be true...),

If subject gets 6 out 8 trials right ($X = 6$):

* probability of actual event or *more extreme* events: $P = \mathrm{P}(X \geq 6) = \mathrm{P}(X{=}6) + \mathrm{P}(X{=}7) + \mathrm{P}(X{=}8) = 0.14$,
* by not too low P-value, observed $X{=}6$ does not seem unreasonable under H_0 (might have happened by chance) $\Rightarrow$ *cannot reject* H_0, (but H_0 could be false...).

Components of a Statistical Test

Statistical Model – main examples so far:
$X \sim B(n, p)$, and $X_1, \ldots, X_n$ i.i.d. (SRS) of population (μ, σ).

Statistical Hypothesis:

- statement/assertion about the model (one or more parameters of the model) which is either true or false,
- null hypothesis H_0 — the one investigated,
- alternative hypothesis H_a — the one to hold if H_0 is not true.

Statistical Test statistic (or test variable):

- "measures" how well the data correspond to H_0 compared to H_a.

P-value (or significance probability):

- the probability, computed under H_0 (assuming H_0 is true), that the test statistic takes a value as extreme as or more extreme than (in the direction of H_a) the actually observed value from the data,[3]
- low P-values provide evidence against H_0
  $\Rightarrow$ rejection of H_0 (and acceptance of H_a, *strong conclusion*),[4]
- high P-values provide no (convincing) evidence against H_0
  $\Rightarrow$ H_0 cannot be rejected (*weak conclusion*).

Significance level α:

- *artificial* borderline/cut-off set *for convenience* between significant (i.e., $P \leq \alpha$) and non-significant (i.e., $P > \alpha$) results,[5]
- *by convention* set at 0.05, or less commonly at 0.10, 0.01, etc.

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[3] The P-value expresses how surprising the observed outcome would be if H_0 was true.

[4] "If the P-value is low, the null hypothesis must go." (Keith Bower; media links)

[5] No uniform rule exists for whether $(P = \alpha)$ is considered significant or not.

Test for Population Mean

Setting for test of population mean:

- Model: $X_1, \ldots, X_n$ i.i.d. from distribution (μ, σ),
  - ∗ assume (approximate) normal distribution of $\bar{X}$,
  - ∗ assume σ known (in practice, rarely a reasonable assumption).
- Null Hypothesis H_0: $\mu = \mu_0$,
  where μ_0 is a known, fixed value (very often, μ_0=0),
- Alternative Hypothesis H_a: $\mu \neq \mu_0$,
- z test statistic computed as

$$z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \quad \sim \quad \text{N(0,1) under } H_0,$$

- P-value computed as

$$P = 2 \times \mathrm{P}(Z \geq |z|) = 2 \times \mathrm{P}(Z \leq -|z|).$$

Example II: Laboratory analysis,

- Data: X_1, X_2, X_3; n=3, $\bar{X}$=0.8404, σ=0.0068 known,
- Hypotheses: H_0: $\mu = 0.86$, H_a: $\mu \neq 0.86$,
- Test statistic: $z = (0.8404 - 0.86)/(0.0068/\sqrt{3}) = -4.98$,
- P-value: $P = 2 \times \mathrm{P}(Z < -4.98) < 2 \times 0.0002 = 0.0004$,
- Conclusion: reject H_0 and accept H_a; strong indication that the specimen content is not as specified (i.e., lower).

One- or Two-sided?

Null hypothesis H_0 usually of the form: parameter=value (e.g. in the laboratory example: $\mu = 0.86$).

Alternative hypothesis H_a usually one of 3 types:

- one-sided upwards: parameter>value (e.g., $\mu > 0.86$),
- one-sided downwards: parameter<value (e.g., $\mu < 0.86$),
- two-sided: parameter different from value (e.g., $\mu \neq 0.86$).

Choice of alternative hypothesis:

- one-sided: when *focus is on particular alternative* (because other direction is difficult to interpret or in beforehand of no interest),
- two-sided: *most common*, when no particular alternative is in focus or no knowledge is present in beforehand.

Alternative hypothesis affects calculation of P-values!

- in general, P-value is probability of extreme events for H_0 relative to (i.e., in the direction of) H_a,
- example: testing for population mean, H_0: $\mu = \mu_0$,

$$\begin{aligned} H_a : \mu > \mu_0 : &\quad P = P(Z \geq z), \\ H_a : \mu < \mu_0 : &\quad P = P(Z \leq z), \\ H_a : \mu \neq \mu_0 : &\quad P = P(Z \geq |z|) + P(Z \leq -|z|). \end{aligned}$$

- P-values and tests may also be termed one/two-sided.[6]

[6] My recommendation is to only talk about one/two-sided alternative hypotheses.

Testing by Confidence Interval

Fact: A confidence interval (CI) for a parameter with confidence level $C = 1-\alpha$ can be used for a significance test at level α for the null hypothesis

$\quad\quad H_0$: parameter = value,

against the alternative hypothesis

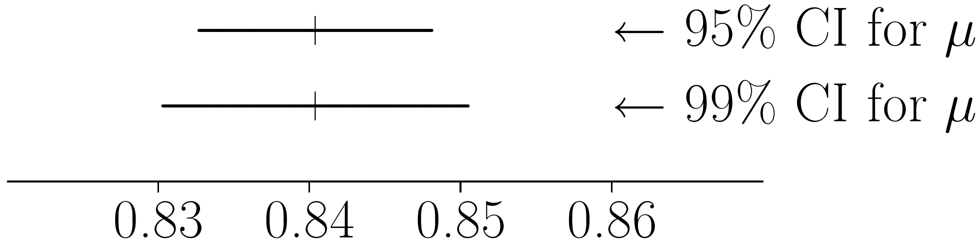
$\quad\quad H_a$: parameter $\neq$ value,

by the following "recipe":

- reject H_0, if value is outside interval.
- cannot reject H_0, if value is inside interval,

Example II: Laboratory analysis,

- 95% CI for μ: $\bar{X} \pm 1.96\,\sigma/\sqrt{3} = 0.8404 \pm 0.0077$,
  $\Rightarrow H_0$: $\mu = 0.86$, rejected at 5% level (also H_0: $\mu = 0.85$),
- 99% CI for μ: $\bar{X} \pm 2.576\,\sigma/\sqrt{3} = 0.8404 \pm 0.0101$,
  $\Rightarrow H_0$: $\mu = 0.86$, rejected at 1% level (but not H_0: $\mu = 0.85$).



Advantages and disadvantages of testing by use of CI:

+ easy (when CI done), enhances CI interpretation,

− no P-value.

Pitfalls of Statistical Testing

Some points to remember when using statistical tests[7]:

- the test/P-value is only as good as its assumptions…,
- strictly speaking, the theory of statistical tests is for hypotheses determined in advance of data collection, and certainly *not inspired by the data…*; also, carrying out many tests on the same data by pure chance may cause some of them to be statistically significant (the multiple testing problem),[8]
- to consider an analysis with $P = 0.049$ a success and discard an analysis when $P = 0.051$, is ridiculous…
(do not put too strong emphasis on significance levels, and always report P-values instead of just significance yes/no),
- non-significance does not mean proof of no effect:
absence of evidence is not evidence of absence! [9]
- non-significance may be important in itself (when not caused by insufficient or otherwise poor data),
- statistical significance does not imply biological significance or causation.

The bottom line is[10]: Statistical testing is often given too much attention in applied data analysis, where one should instead focus on the estimates, their precision (indicated e.g. by a confidence interval) and their interpretation, plus many other decisions prior to the final results.

[7] Based on PSLS, Chapter 15, and IPS, Section 6.3.

[8] Computing 20 tests, each with 5% error rate ⇒ 5% error rate? – or $5 \times 20\% = 100\%$ error rate? — the correct answer is in-between.

[9] Quote usually attributed to Carl Sagan; or to William Cowper, 1731-1800.

[10] You can find published articles recommending significance tests to be abolished, but this *is not* the current consensus; see e.g. homepage media links.

1-Sample Estimation

Data: sample $X_1, \ldots, X_n$ of size n from some distribution with unknown mean μ and unknown standard deviation σ (and variance σ^2). More specifically, we assume

- the X's are i.i.d. (independent, identically distributed),
- $\mathrm{E}X_i = \mu$ and $\mathrm{sd}X_i = \sigma$ for all X's.

For estimation of σ we use the sample standard deviation:

$$\hat{\sigma} = s \ (= \sqrt{s^2}) \quad \text{and} \quad \hat{\sigma}^2 = s^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1),$$

and s^2 is an *unbiased* estimate of σ^2. This explains our use of $(n-1)$ in the denominator of s^2. [11]

Summary of terminology and estimates for a single sample:

name	estimate	parameter	properties
sample mean	$\bar{X}$	μ	unbiased
sample variance	s^2	σ^2	unbiased
sample standard deviation	s	σ	biased, natural
(sample variance of mean)	s^2/n	$\sigma_{\bar{X}}^2 = \sigma^2/n$	unbiased
standard error of mean[1]	$s/\sqrt{n}$	$\sigma_{\bar{X}} = \sigma/\sqrt{n}$	biased, natural

[1] abbreviations: SE, $\mathrm{SE}_{\bar{X}}$, SEM, s.e., …

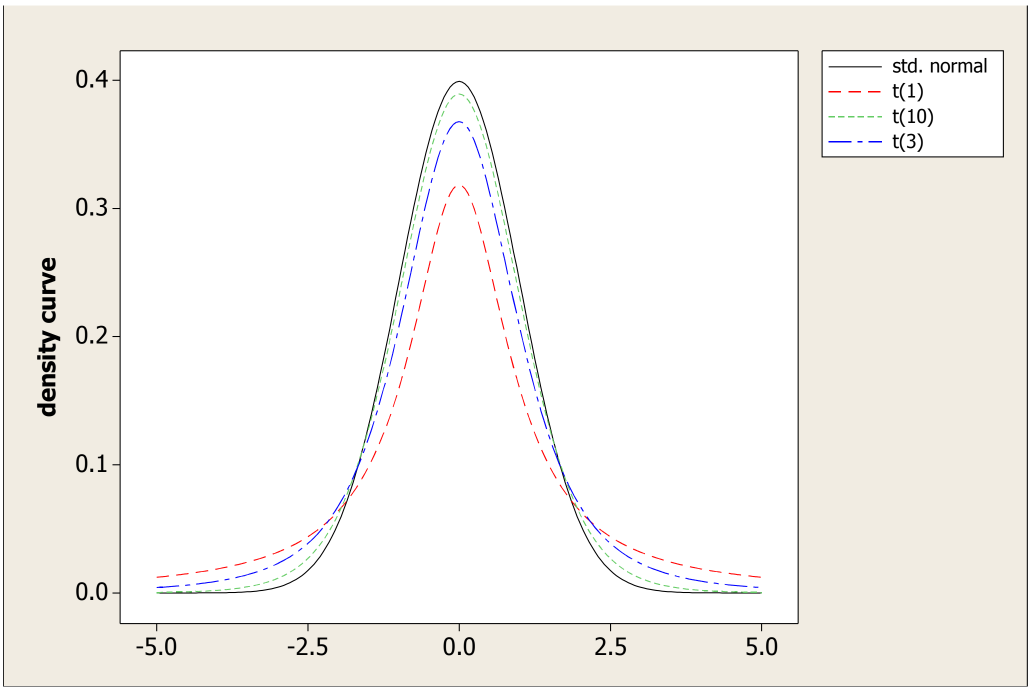
In addition, $\bar{X} \sim \mathrm{N}(\mu, \sigma_{\bar{X}})$

- exactly — if the X's are normally distributed,
- approximately (when n is "large") — always! (by CLT)

[11] We skip over the (small) mathematical calculation showing that s^2 is unbiased.

(STUDENT'S) t DISTRIBUTION

- a new distribution — to be used for inference in a normal model when σ is estimated from the data:
  — reference distribution for t test statistics,
- has a single parameter r (or "df"):
  - $r = 1, 2, 3, \ldots$
  - called "degrees of freedom" (explanation to follow),
  - given from the data, and not to be estimated.
- denoted $t(r)$ to indicate degrees of freedom,
- distribution on $(-\infty, \infty) \Rightarrow$ positive and negative values,
- symmetric around zero, almost "bell-shaped" but with heavier tails than N(0,1),
- when r large: $t(r) \approx$ N(0,1).



1-Sample Normal Distribution Inference

- Data: $X_1, \ldots, X_n$ (n = number of observations).
- Model: observations are a sample (i.i.d.) from $\mathrm{N}(\mu, \sigma)$, where μ and σ are unknown parameters.
- Estimation: $\hat{\mu} = \bar{X}$ and $\hat{\sigma} = s$.
- Distribution of estimates:

$$\hat{\mu} = \bar{X} \sim \mathrm{N}(\mu, \sigma/\sqrt{n}), \quad s_{\bar{X}} = s/\sqrt{n},$$
$$(\bar{X} - \mu)/s_{\bar{X}} \sim t(n-1),$$

  note that degrees of freedom (df) $= n-1$,
- Confidence interval with confidence level $1-\alpha$:

$$\mu : \bar{X} \pm t^* s_{\bar{X}} = \bar{X} \pm t^* s/\sqrt{n},$$

  where t^* is a suitable value from a $t(n-1)$ distribution (→ Table C of PSLS, Table 3 of S, Table D of IPS),[12]
- Test of H_0: $\mu = \mu_0$ against alternative H_a:
  * test statistic: $t = (\bar{X} - \mu_0)/s_{\bar{X}}$,
  * P-value from t distribution with df $= n-1$:
    - H_a: $\mu \neq \mu_0$: $P = 2 \times \mathrm{P}(t(\mathrm{df}) \geq |t_{\mathrm{obs}}|)$,
    - H_a: $\mu > \mu_0$: $P = \mathrm{P}(t(\mathrm{df}) \geq t_{\mathrm{obs}})$,
    - H_a: $\mu < \mu_0$: $P = \mathrm{P}(t(\mathrm{df}) \leq t_{\mathrm{obs}})$,
- note strong similarities with z-based procedures.

[12] Specifically, $t^* = t_{1-\alpha/2}(n-1)$ is the $(1-\frac{\alpha}{2})$-percentile of a $t(n-1)$ distribution.

Example: Human Body Temperature

Example 14.9 of PSLS 3e, Example p. 139 of S:

- Data: 130 measurements[13] of body temperature in °F of healthy adults: $X_1, \ldots, X_{130}$ ($n = 130$).
- Model: a sample (i.i.d.) from $\mathrm{N}(\mu, \sigma)$.
- Estimation: $\hat{\mu} = \bar{X} = 98.25$ and $\hat{\sigma} = s = 0.733$.
- Confidence interval with confidence level 95% ($\alpha = 0.05$):

$$\begin{aligned} \mu \ : \ & \bar{X} \pm t^* s_{\bar{X}} = 98.25 \pm 1.98 \times 0.733/\sqrt{130} \\ = \ & 98.25 \pm 0.13 = (98.12, 98.38), \\ & (t^* = t_{.975}(129) = 1.9785 \text{ from Minitab}), \end{aligned}$$

- Test of H_0: $\mu = 98.6$ against alternative H_a: $\mu \neq 98.6$: ("classical" average body temperature)
  - test statistic: $t = \dfrac{\bar{X} - 98.6}{s_{\bar{X}}} = \dfrac{98.25 - 98.6}{0.733/\sqrt{130}} = -5.45$,
  - P-value from t distribution with $\mathrm{df} = n-1 = 129$:

$$P = 2 \times \mathrm{P}(t(129) \geq 5.45) < 0.000001 \text{ (Minitab)}$$

  - Conclusion: strong evidence to say that average body temperature is different, actually lower, than "classical" reference value.

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[13] Constructed data for pedagogical purposes (Shoemaker (1996), *Journal Statistics Education* **4**) based on a real study from 1992 whose purpose it was to evaluate the well-established average body temperature of 37.0°C or 98.6°F, Mackowiak et al. (1992), *Journal of the American Medical Association* **268**, 1578-1580.

How to Find Percentiles and P-values

Recall that

- the $p\%$ percentile has $p\%$ of the distribution below, and $100-p\%$ above, where $100-p\%$ is the tail probability,[14]
- P-values are typically determined as tail probabilities in standard distributions like N(0,1) or t(df).

Methods to determine percentiles or tail probabilities:

- Minitab: `Inverse Cumulative Probability` for percentiles and `Cumulative Probability` for probabilities like $\mathrm{P}(t \leq t_{\text{obs}})$ (1−tail probabilities),
- Stata: functions `normal`, `invnormal`, `ttail`, `invttail`,
- R: functions `pnorm`, `qnorm`, `pt`, `qt`,
- statistical tables: values for some confid./error levels.

What to do if df is not in table:

- use largest value below df
  ⇒ conservative analysis (larger CI's and P-values).

What to do if t_{obs}-value is not in table?

- find closest ("critical") values in table, for example $t_1 < t_{\text{obs}} < t_2$, and use the relations

$$\mathrm{P}(t \geq t_2) < \mathrm{P}(t \geq t_{\text{obs}}) < \mathrm{P}(t \geq t_1).$$

[14] In statistical tables, the $p\%$ percentile is the critical value for a one-tailed $\alpha = 100-p\%$.

Exercises 6.47, Extra, and 7.50

Exercise 6.47:
Null and alternative hypotheses for testing problems:

(a) H_0: $\mu = 18$ and H_a: $\mu < 18$,

(b) H_0: $\mu = 50$ and H_a: $\mu > 50$,

(c) H_0: $\mu = 24$ and H_a: $\mu \neq 24$.

Extra Exercise ($\sim$ 6.77 of IPS7e):
Explain in simple language why a test significant at the 1% level is also significant at the 5% level.
Some possible explanations:

- a P-value below 1% is also below 5%,
- an event occurring by chance with probability less than 1% also occurs with prob. less than 5%,
- it is "more difficult" (stronger requirement) to be significant at 1% than 5% level.

Exercise 7.50:
Percentiles/critical values for confidence intervals for population mean (with unknown population stand. deviation):

(a) $n = 20$ and $C = 95\%$:
$\alpha = 0.05$ and $t^* = t_{1-\alpha/2}(n-1) = t_{.975}(19) = 2.093$.

(b) $n = 30$ and $C = 90\%$:
$\alpha = 0.10$ and $t^* = t_{1-\alpha/2}(n-1) = t_{.95}(29) = 1.699$.

(c) $n = 50$ and $C = 80\%$:
$\alpha = 0.20$ and $t^* = t_{1-\alpha/2}(n-1) = t_{.90}(49) \approx t_{.90}(40) = 1.303$,
— a conservative value (exact value (software): 1.299).

Summary Notes

Key words and concepts:

- statistical test:
  - concepts: null hypothesis H_0, alternative hypothesis H_a (one or two-sided), test statistic and its (reference) distribution, P-value, significance level,
  - possible conclusions: reject H_0 (and favour H_a), or no (insufficient) evidence against H_0,
  - z-test formula for mean in normal distrib. (with known σ),
- relation between test and confidence interval (for single param.),
- common misconceptions related to statistical testing (see 6L–8),
- statistical inference for 1 sample on normal distrib. (unknown μ, σ):
  - sample standard deviation (s) as estimate of population standard deviation (σ), standard error (for sample mean),
  - t-distribution, degrees of freedom,
  - formulae for t-based confidence interval and t-test,
- finding/approximating P-values and critical values (t^*).

Four-step process for tests (PSLS 3e):

State: What is the practical question that requires a statistical test?

Plan: Identify a parameter, state the null and alternative hypotheses, and choose the type of test that fits your situation.

Solve: Carry out the test in three phases:

- Check the conditions for the test you plan to use.
- Calculate the test statistic.
- Find the P-value using a table of Normal probabilities or technology.

Conclude: Return to the practical question to describe your results in this setting.