## Supplementary exercises 7.73 and 7.74 of IPS7e

Phosphorus levels ($mg/dl$) in the blood of one patient, measured at $n = 6$ occasions.

Data: $X_1, \ldots, X_6$.

Model: $X_1, \ldots, X_6$ are i.i.d. (a simple random sample, SRS) and normally distributed $\mathrm{N}(\mu, \sigma)$, where $\mu$ and $\sigma$ are unknown parameters, corresponding to the values of this patient (note: not a wider population of patients).

*Exercise 7.73*

(a) We estimate the parameters as follows,

$$\begin{aligned} \hat{\mu} &= \bar{X} = 5.367, \\ \hat{\sigma} &= s = 0.6653, \quad \text{and therefore,} \\ \mathrm{SE} &= \widehat{\mathrm{sd}}(\bar{X}) = s/\sqrt{6} = 0.2716. \end{aligned}$$

(b) With a confidence level of 90%, our $t^*$-value is the 95% percentile of $t(5)$, the $t$-distribution with 5 df, which equals $t^* = 2.015$. Then

$$90\%\ \mathrm{CI}: \ \bar{X} \pm t^* s/\sqrt{n} = 5.367 \pm 2.015 \cdot 0.2716 = 5.367 \pm 0.547 = (4.82, 5.91).$$

For calculation in Minitab, we give the command (corresponding to the `Basic Stat-1 sample t` menu) and output:

```
OneT 'phos';
 Confidence 90;
 Alternative 0.
```

One-Sample T: phos

Descriptive Statistics

| N | Mean | StDev | SE Mean | 90% CI for μ |
|---|---|---|---|---|
| 6 | 5.367 | 0.665 | 0.272 | (4.819, 5.914) |

μ: mean of phos

*Exercise 7.74*

The normal range for phosphorus levels is $2.6 - 4.8$. In order to assess whether the data indicate an elevated phosphorus level, we should test

$$\mathrm{H}_0 : \mu = 4.8 \quad \text{versus} \quad \mathrm{H}_a : \mu > 4.8.$$

Note that the question asked refers to a phosphorus level exceeding 4.8, however with the potentially invalid justification that the patient's mean level is high. It is **not** valid to look at the data and decide the alternative based on the observed values! There may however be valid reasons to be particularly concerned about elevated phophorus levels.

Our $t$-test statistic is,

$$t = (\bar{X} - 4.8)/\mathrm{SE} = (5.367 - 4.8)/0.2716 = 2.088.$$

A $t$-distribution table (Table C of PSLS, Table 3 of S, Table D of IPS) gives the percentiles of $t(5)$ as:

$$95\%\ \text{percentile} = 2.015, \quad 97.5\%\ \text{percentile} = 2.571.$$

In the table, these are labeled as "critical values" for 1 minus the one-sided (or one-tailed) $\alpha$ (or $P$); the corresponding values are 0.05 and 0.025, respectively. The observed $t$-value falls between

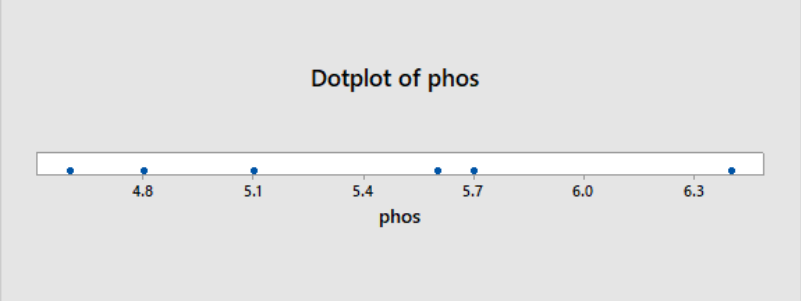
these two limits, therefore the $P$-value is between 0.025 and 0.05 (and presumably closest to 0.05). Statistical software gives $P = 0.046$. Formally, this constitutes evidence (at the 5% level) that the patient's phosphorus is higher than normal, however because the $P$-value is so close to 0.05 we should consider this as weak evidence only. In practice the strength of evidence required will often depend on the consequences. For example, if the follow-up procedure is a more detailed examination of the patient, no further evidence may be required.

The assumptions of the analysis are:

*i)* independent observations,
*ii)* a normal distribution of the phosphorus measurements (and for such a small sample the robustness of the $t$-procedure against violations is not great, so some justification of the normality assumption is needed),
*iii)* same mean and standard deviation of all observations.

The small sample size means that it is almost pointless to try assess the normal distribution from the data. The best descriptive display of the data is probably a dotplot, as shown below. The justification of assuming a normal distribution should come from knowledge about the distribution of phosphorus measurements in a larger sample.

Also here we include the Minitab command and output from the $t$-test menu.




```
OneT 'phos';
 Test 4.8;
 Confidence 90;
 Alternative 1.
```

One-Sample T: phos

Descriptive Statistics

| N | Mean | StDev | SE Mean | 90% Lower Bound for μ |
|---|---|---|---|---|
| 6 | 5.367 | 0.665 | 0.272 | 4.966 |

*μ: mean of phos*

Test

Null hypothesis $H_0$: $\mu = 4.8$
Alternative hypothesis $H_1$: $\mu > 4.8$

| T-Value | P-Value |
|---|---|
| 2.09 | 0.046 |

**Technical note**:
The last Minitab listing with the $t$-test against a one-sided alternative includes a one-sided CI instead of the usual two-sided CI, similarly to what we noted for the $z$-test menu/output (see e.g. Exercise 6.140). In VHM 801 and also in general statistical practice, all confidence intervals are two-sided. One practical implication of this Minitab setting is therefore that if one wants both a confidence interval and a test against a one-sided alternative, the $t$-test menu needs to be run twice.