

Index of Lecture 10a

(with a few updates)

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PRACTICAL INFORMATION

Today's lecture:

- introduction to clustered data (Javier),
- linear mixed models (Henrik/these notes):
 - * introduction focusing on main principles,
 - VHM 802 (not offered this year),
 - multilevel summer course (June 2020; more information at <http://cver.upei.ca>).

Textbook (VER2/MER) reading:

- Clustered data: Sections 20.1-3 with some skippings,
- Linear mixed models: Sections 21.1-2 and parts of 21.5,

Home work for Tuesday next week (VHM 812 review session):

- sample problem for VER Chapter 20 (and 21) using dataset `ap2_intro`: Questions 1-4 (Question 5 is optional).

MIXED MODELS FOR CONTINUOUS DATA

(INCL. VARIANCE COMPONENT ESTIMATION)

Synthesis:

- mixed models extend ordinary linear models (regression and ANOVA) to take into account “clustering”.

Contents:

- introduction to mixed models and modelling
 - * theory (gently), notation and practice,
 - * brief overview of main modelling steps,
- Stata computer demonstrations.

Terminology and relationships:

- mixed
random effects
variance component } models – the same,
- “mixed” \sim containing both fixed and random effects,
- multi-level
hierarchical } models – the same,
and special type of mixed models,
- variance components are mathematical constructs used in mixed models.

CONCEPTUAL EXAMPLE

Consider the following problem:

- study of risk factors for (high) somatic cell counts (e.g., as a crude indicator of mastitis),
- one recording (for simplicity) of the cell count in a milk sample from each cow; in total, n cows,
- additional recordings of explanatory variables for each cow, such as lactation stage (days in milk), age, breed,...
- also explanatory variables at the herd level, e.g. housing type,
- linear model:

$$Y_i = \beta_0 + \beta_1 x_{1i} + \dots + \beta_k x_{ki} + \varepsilon_i, \quad (1)$$

or

$$Y = X'\beta + \varepsilon,$$

where

- * Y_i = (natural) log somatic cell count for cow i ,
- * x_{ri} 's contain values of the explanatory variables,¹
- * $(\beta_0), \beta_1, \dots, \beta_k$ are regression coefficients for x 's,
- * ε_i = error term $\sim N(0, \sigma^2)$,
- * i = cow number: $1, \dots, n$.

¹ In this notation (from the VER2 textbook), we use x_{ri} instead of the usual x_{ir} ; $X' = (x_{ri})_{ir}$ is the $n \times (k+1)$ design matrix, including as (x_{0i}) a column of 1's.

MIXED MODEL FOR HIERARCHICAL DATA

Simplest case \sim extended cell count example:

assume measurements on cows in several herds,

$$Y_{ij} = \beta_0 + \beta_1 x_{1ij} + \dots + \beta_k x_{kij} + u_j + \varepsilon_{ij}, \quad (2)$$

or

$$Y = X'\beta + u + \varepsilon,$$

where

- Y_{ij} = log somatic cell count for cow i in herd j ,
- u_j = random j^{th} herd effect $\sim N(0, \sigma_h^2)$,
- σ_h = scale of random herd effects, interpretable as the amount of random variation in log-scc between herds;
 - * e.g., 95% of herds expected within $0 \pm 1.96 \sigma_h$,
- i = cow number, j = herd number.

Definitions (non-Bayesian terminology):

- “random” effect: a model term (right hand side) which is a random variable (often not counting the error term ε),
- “fixed” effect: modelled by non-stochastic parameters (β 's, often not counting the intercept β_0),
- mixed model: both fixed and random effects.

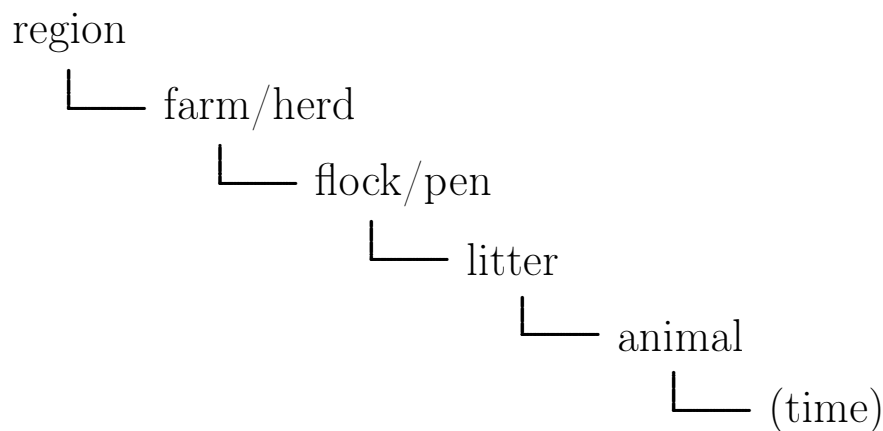
HIERARCHICAL DATA STRUCTURE

A hierarchical data structure:

- observations grouped at different levels,
- treatments applied at different levels.

may induce clustering in the data, that is, some observations are more alike than others, or put in another way: the observations are no longer independent.

Typical example from veterinary epidemiology:



Note: “time” as a bottom level \sim longitudinal data / repeated measures on the same animal, and raises some additional modelling issues.

Random effects models for hierarchical data:

- insert random effect(s) for each hierarchical level (above the bottom level).

RANDOM EFFECTS

- the only new concept in mixed models,
- enable a separation (and quantification) of variation at different levels in the data,
- enable correct analysis of predictors at different levels within the same model,
- involve additional assumption(s) of normal distribution and variance homogeneity.

Motivations for random effects (decreasing importance):

- hierarchical data structure:
 - * rule: insert a random effect for each hierarchical level above the bottom level (exceptions: L10a–13),
- correct analysis of treatments allocated to larger experimental units (“split-plot” idea²),
- factor where interest is in the variation between units (within a level) rather than specific units in study:
 - * units may be randomly selected,
 - * units should represent “population” — to which the conclusions from the study may be generalized,
- avoid many (nuisance³) parameters in model/estimation.

² Split-plot designs are experimental designs where treatment factors (deliberately) are applied to units of different sizes; discussed in detail in VHM 802.

³ Of no or little intrinsic interest.

SOMATIC CELL COUNT DATASETS

scc_40 – a real somatic cell count dataset:

A subset including 40 herds from a large dataset collected in 1993-94 by J. Agger and co-workers including about 2150 Danish herds and 150 000 cows followed throughout one lactation. The data contain approx. monthly milk records plus information collected through herd questionnaires.

Variable	Description	Values
herdid	herd id	1 – 40
cowid	cow id	1 – 2178
test	approx. month of lactation	0 – 10
t_lnscc	natural log scc (in 1000s) on test day	2.3 – 9.2
t_dim	days in milk on test day	10 – 305
t_season	season of test day	1 – 4 (1 = Jan-Mar, etc.)
c_heifer	parity of cow	0/1 (1 = heifer)
h_size	average herd size	10.3 – 101.5
t_ecm	energy-corrected ¹ milk yield	2.2 – 68.5

¹ computed by the formula:

$$\text{ecm} = \text{kgmilk}(0.383 \text{ fatpct} + 0.242 \text{ proteinpct} + 0.7832)/3.14.$$

Subdataset scc40_2level:

only first observation per lactation included

⇒ one observation per cow.

MIXED MODEL FOR SOMATIC CELL COUNT DATA

- data: 2-level somatic cell count data (`scc40_2level`),
- outcome: log somatic cell count (`t_lnscc`),
- fixed effects: season (categ.), dim, heifer, hsize (all quant.),
- random effects: herds (because only 1 obs. per cow).

```
. mixed t_lnscc h_size c_heifer i.t_season t_dim || herdid:, reml
```

```
Mixed-effects REML regression          Number of obs      =      2178
Group variable: herdid                 Number of groups   =         40

                                      Obs per group: min =         12
                                      avg =          54.5
                                      max =          105

                                      Wald chi2(6)       =      244.36
                                      Prob > chi2        =      0.0000

Log restricted-likelihood = -3624.9622
```

t_lnscc	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
h_size	.0040837	.0037726	1.08	0.279	-.0033105	.0114778
c_heifer	-.7367168	.0554447	-13.29	0.000	-.8453863	-.6280472
t_season						
apr-jun	.1609431	.0906574	1.78	0.076	-.0167422	.3386285
jul-sep	.0015031	.0863774	0.02	0.986	-.1677935	.1707997
oct-dec	.0014582	.0918936	0.02	0.987	-.1786499	.1815663
t_dim	.0027731	.0004991	5.56	0.000	.0017949	.0037513
_cons	4.641202	.1974215	23.51	0.000	4.254263	5.028141

Random-effects Parameters	Estimate	Std. Err.	[95% Conf. Interval]	
herdid: Identity				
var(_cons)	.1491533	.0436191	.0840821	.2645832
var(Residual)	1.557228	.0477206	1.466451	1.653625

```
LR test vs. linear regression: chibar2(01) = 97.01 Prob >= chibar2 = 0.0000
```

REUNION ISLAND STUDY

- carried out 1993-1996 on Reunion Island,
- data analyzed 1996-2000, and results communicated to the cattle industry and published 1999-2001.

Objective:

identify the factors and levels at which most of the variability in reproductive performance resides ...
where interventions are likely to have the most effect.

Reproductive performance in cows measured by time from calving to conception, which is composed of

- time from calving to first service,
- conception from first service (yes/no),
- if no, time from first service to conception.

Data size and structure:

Level	Number	Per unit at level above	
		Average	Range
Region	5	—	—
Herd	50	10.0	3–16
Cow	1575	31.5	8–105
Lactation	3027	1.9	1–5

— no cow movements between herds (~ strict hierarchical structure).

VARIANCE COMPONENTS

Always a decomposition of the total variation:

$$\text{total variation} = \text{fixed effects var.} + \text{random var.}^4$$

Additionally in random effects models:

– a decomposition of the random variation.

2-level model – somatic cell count example (herds–cows):

- $\text{Var}(Y_{ij}) = \sigma^2 + \sigma_h^2$ (= total unexplained random var.),
- variance components σ^2 and σ_h^2 ,
- of the total random variation, $\sigma_h^2/(\sigma^2 + \sigma_h^2)$ resides at the herd level, and the rest at the cow level,⁵
- ICC⁶ = $\sigma_h^2/(\sigma^2 + \sigma_h^2)$, often denoted ρ (as a corr. coef.).

Multilevel models – Reunion ex. (herds–cows–lact.):

- $\text{Var}(Y_{ij}) = \sigma^2 + \sigma_c^2 + \sigma_h^2$,
- proportions of variance at diff. levels in the obvious way,
- two ICCs:

$$\text{lactations of same cow} : (\sigma_c^2 + \sigma_h^2)/(\sigma^2 + \sigma_c^2 + \sigma_h^2),$$

$$\text{lactations in same herd} : \sigma_h^2/(\sigma^2 + \sigma_c^2 + \sigma_h^2),$$

general formula: sum of variance components of common random effects divided by sum of all variance comp.

⁴ Least squares decomposition: $\sum(Y_i - \bar{Y})^2 = \sum((X'\hat{\beta})_i - \bar{Y})^2 + \sum(Y_i - (X'\hat{\beta})_i)^2$

⁵ Also termed variance partition coefficient (VPC).

⁶ Intra-class (or -cluster) correlation coefficient: the correlation between two obs. in the same class/cluster. Alternative *interpretations of clustering* as: variation between clusters, or correlation within clusters.

ESTIMATION IN LINEAR MIXED MODELS

“Likelihood”-based estimation assuming normal distributions for all random terms:

- REML (restricted maximum likelihood) or (full) ML:
 - * theoretical properties differ slightly (REML unbiased, ML less variance),
 - * in practice only minor differences, unless the number of units at a level is small,
 - * REML estimates agree with ANOVA-type⁷ estimates for balanced data,
- iterative, numerically robust algorithms,
- performs well for both balanced and unbalanced data,
- available in many statistical software packages (also in Minitab 18+: `Mixed Effects Model` menu), however
 - * different modelling flexibility,
 - * different ability to handle large data structures,
- should give *the same* estimates from different software packages, up to estimation accuracy, despite minor differences in implementations.

⁷ A classical statistical method for variance component models relies on the ANOVA-table, and constructs estimates and test statistics from the MS-column; performs well only for almost balanced data.

STATISTICAL INFERENCE IN LINEAR MIXED MODELS

Tests and confidence intervals, approximate and assuming normal distributions for all random terms:

- Wald statistics for fixed effects: based on standard errors and estimated correlations between estimates,
 - * 95% confidence intervals: $\hat{\beta}_r \pm 1.96 \times \text{SE}(\hat{\beta}_r)$,
(better to use t -distribution percentile with suitable df⁸, but similar if df is large),
 - * simple to compute, for fixed effect parameters usually ok,

- likelihood-ratio tests, based on optimal values of likelihood function (differences of $-2 \log L$):⁹

$$G^2 = 2(\log L_{\text{full}} - \log L_{\text{red}}) \sim \chi^2(\text{df}),$$

where df = no. of parameters being tested equal to zero,

- * the only appropriate test for variance parameters,¹⁰
- confidence intervals for variance parameters: not easy; usually ok to present approximate intervals in Stata from Wald-type procedure (but inference from these can be misleading).

⁸ In Stata 14+, options `dfmethod(satterthwaite)` or `dfmethod(kroger)`.

⁹ Caution for fixed parameters and REML estimation: beware to *not* use the restricted likelihood. Stata's `lrtest` command gives a warning note.

¹⁰ Note: the P -value should be *half* the value from $\chi^2(\text{df})$ when testing $H_0 : \sigma_h^2 = 0$. (Stata does this per default.)

STATISTICAL INFERENCE (CONT.)

Model-building guidelines:

- fixed effects: similar to linear models,¹¹
- random effects: generally one for each hierarchical level, but some exceptions:
 - * fixed effects possible, if no. of units is small and/or unrepresentative of a population, and no predictor has variation at that level (typically at highest level),
 - * level may need to be omitted if only very few replications at that level present in the data.

Model checking:

- now model assumptions (and potential violations) at multiple levels,
- predictions, residuals and diagnostics at multiple levels,
- software differences in the accessibility of these statistics; Stata gives easy access to
 - * lowest level residuals (not standardized well) + fitted values \Rightarrow usual model checking at lowest level,
 - * predicted random effects (“BLUPs”) at higher levels \Rightarrow normality checks can be done easily, and plots against predicted values with some work (do-file),
- model checking is an informal, exploratory process.

¹¹ Fixed effects model-building should use models with all relevant random effects.